THREE-PAGE EMBEDDINGS OF SINGULAR KNOTS

V. KURLIN* AND V. VERSHININ**

Abstract. Construction of a semigroup with 15 generators and 84 relations is given. The center of this semigroup is in one-to-one correspondence with the set of all isotopy classes of non-oriented singular knots (links with finitely many double intersections in general position) in $\mathbb{R}^3$.

1. Introduction

1.1. **Statement of the problem and results.** We develop the Dynnikov method of three-page embeddings for *links with singularities* of the following type: finitely many double intersections in general position are possible. More precisely, an algebraic solution of isotopy classification problem for non-oriented singular knots in $\mathbb{R}^3$ is given. The key idea is a construction of a 3-page embedding for a neighborhood of a singular point. In particular this construction gave a possibility of diminishing the number of generators and defining relations in the semigroup for singular knots.

1.2. **Review of the previous results.** An embedding of a link in a structure which looks like an open book with finitely many pages probably was considered for the first time by H. Brunn in 1898 [3]. Namely, he proved that each knot is isotopic to a knot that projects to a plane with only one singular point. Later such embeddings were studied in papers [4, 5, 8] and were used in [6]. Moreover, these investigations provided a new link invariant – *arc index* [7, 20, 21]. It turned out that each link embeds in a book with only 3 pages. In 1999 Dynnikov classified all non-oriented links in $\mathbb{R}^3$ up to an ambient isotopy encoding them by *three-page diagrams* [10, 11]. To be more precise we call these diagrams *3-page embeddings* (see the definition in Subsection 2.1). Dynnikov constructed some semigroup DS such that there is a one-to-one correspondence between the center of DS and the set of all isotopy classes of non-oriented links in $\mathbb{R}^3$. Applying embeddings into a book with an arbitrary number of pages Dynnikov decreased the number of the relations in his semigroup DS [12]. Analogously the first author obtained an isotopic classification of non-oriented knotted 3-valent graphs in $\mathbb{R}^3$ [19].

1.3. **Motivations.** In [16] singular knots were called *chimerical graphs* and in [17] four-valent graphs *with rigid vertices*. The recent study of singular knots and braids was motivated by the theory of Vassiliev invariants [2]. The corresponding algebraic object is called the *Baez-Birman monoid* or *singular braid monoid* [1, 2]. Some of its algebraic properties were investigated in [9, 13, 15]. For singular braids the analogue of Markov's theorem was proved [14]. Many invariants of regular (non-singular) links, in particular, the Alexander-Conway and Jones polynomials and Vassiliev invariants are extended to singular knots [16, 17, 18, 22]. Homological properties of singular braids on infinitely many strings were studied by the second author [24, 25].

1.4. **Basic definitions.** We work in the PL-category, i.e. images of circles under immersion in $\mathbb{R}^3$ are finite polygonal lines. Formally, a *singular knot* is an immersion of several circles into $\mathbb{R}^3$ with possible double intersections in general position in a finite number of *singular points* (Fig. 1, 2). Two *branches* of a given singular knot pass through each singular point. In the present paper (except Subsection 3.5) we consider only non-oriented singular knots, may be non-connected. Note that a singular knot is a 4-valent graph embedded in $\mathbb{R}^3$. An *ambient PL-isotopy* between two graphs is a continuous family of PL-homeomorphisms $\phi_t : \mathbb{R}^3 \to \mathbb{R}^3$, $t \in [0,1]$, such that $\phi_0 = \mathrm{id}$ and ϕ_1 sends

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one of the graphs to another. Singular knots are considered up to ambient PL-isotopy that respects the rigidity (or *template* by the terminology of the work [16]) of each singular point. If we omit the last restriction on isotopy then we come to the notion of *knuckle* 4-valent graph (graph with non-rigid vertices according to the terminology of the work [17]). Contrary to the case of singular knots an isotopy of knuckle graphs can transpose edges at each 4-valent vertex. In Subsection 3.5 we formulate classification Theorem 5 for knuckle graphs. The same as for regular links, one can represent singular knots by plane diagrams equivalent up to the Reidemeister moves $R1-R5$ of Fig. 1 [16]. We depict only PL-analogues of the corresponding smooth moves and omit subdivisions and extra breaks of edges. In the case of knuckle 4-valent graphs the move $R5'$ is taken instead of $R5$.

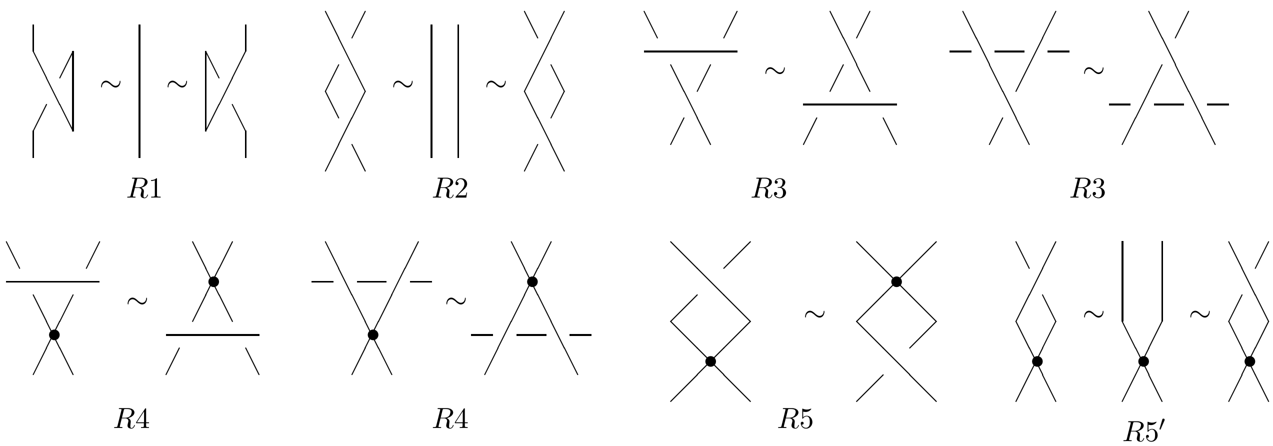

Fig. 1: Reidemeister moves for singular knots and knuckle graphs

1.5. **The universal semigroup for singular knots.** Everywhere the index i belongs to the group $\mathbb{Z}_3 = \{0, 1, 2\}$. Consider the alphabet $\mathbb{A} = \{a_i, b_i, c_i, d_i, x_i \mid i \in \mathbb{Z}_3\}$ with 15 letters (see Fig. 3 in Subsection 2.3 for their geometrical interpretation). Let SK be the semigroup on 15 generators of the alphabet $\mathbb{A}$ and relations (1)-(10), which correspond to some "elementary ambient isotopies" of singular knots in $\mathbb{R}^3$.

(1) $a_i = a_{i+1}d_{i-1}, \quad b_i = a_{i-1}c_{i+1}, \quad c_i = b_{i-1}c_{i+1}, \quad d_i = a_{i+1}c_{i-1},$
(2) $x_i = d_{i+1}x_{i-1}b_{i+1},$
(3) $d_0d_1d_2 = 1,$
(4) $b_id_i = d_ib_i = 1,$
(5) $d_ix_id_i = a_i(d_ix_id_i)c_i, \quad b_ix_ib_i = a_i(b_ix_ib_i)c_i,$
(6) $x_i(d_{i+1}d_id_{i-1}) = (d_{i+1}d_id_{i-1})x_i,$
(7) $(d_ic_i)w = w(d_ic_i),$ where $w \in \{c_{i+1}, x_{i+1}, b_id_{i+1}d_i\},$
(8) $(a_ib_i)w = w(a_ib_i),$ where $w \in \{a_{i+1}, b_{i+1}, c_{i+1}, x_{i+1}, b_id_{i+1}d_i\},$
(9) $t_iw = wt_i,$ where $t_i = b_{i+1}d_{i-1}d_{i+1}b_{i-1}, w \in \{a_i, b_i, c_i, x_i, b_{i-1}d_id_{i-1}\},$
(10) $(d_ix_ib_i)w = w(d_ix_ib_i),$ where $w \in \{a_{i+1}, b_{i+1}, c_{i+1}, x_{i+1}, b_id_{i+1}d_i\}.$

One relation in (4) is superfluous: it can be obtained from (3) and the rest of the relations in (4). Hence the total number of relations (1)-(10) is 84.

1.6. **Algebraic classification of singular knots.**

Theorem 1. *Each singular knot can be represented by an element of the semigroup SK.*

Theorem 2. *Two singular knots are ambiently isotopic in $\mathbb{R}^3$ if and only if the corresponding elements of the semigroup SK are equal.*

Theorem 3. *An arbitrary element of semigroup SK corresponds to a singular knot if and only if this element is central, i.e. it commutes with every element of SK.*

As it will be shown in Theorem 4 the whole semigroup SK describes a wider class of *3-page singular tangles*. The subsemigroup in SK that is generated by 12 letters a_i, b_i, c_i, d_i ($i \in \mathbb{Z}_3$) and 48 relations

from (1)-(10) not containing letters x_i ($i \in \mathbb{Z}_3$) coincides with Dynnikov's semigroup DS of [10, 11]. The center of semigroup DS classifies all non-oriented (regular) links in $\mathbb{R}^3$ up to ambient isotopy.

1.7. **The content of the paper.** In Subsection 2.1 we define three-page embeddings of singular knots. Such embeddings are constructed from plane diagrams of knots in Subsection 2.2 and are encoded in Subsection 2.3. Theorem 1 is proved in Subsection 2.5. The ordinary singular tangles and three-page singular tangles are introduced in Subsection 3.1 and Subsection 3.2, respectively. The later notion generalizes 3-page embeddings of singular knots and helps us in the proof of Theorem 2. In Subsection 3.3 the three-page tangles are classified (Theorem 4). Then Theorem 2 follows from Theorem 4 as a particular case. Theorem 2 is applied in the proof of Theorem 3 in Subsection 3.4. In Subsection 3.5 classification Theorem 5 for knuckle 4-valent graphs is formulated. In Section 4 we deduce Lemma 3 which is used in the proof of Theorem 4.

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2. Three-page embeddings

2.1. **Formal definition of three-page embeddings.** An *arc* of a singular knot $K \subset \mathbb{R}^3$ with endpoint $A \in K$ is a sufficiently small segment $J \subset K$ with $A \in \partial J$. Then 4 arcs issue from each singular point. Let P_0, P_1 and P_2 be three half-planes in $\mathbb{R}^3$ with a common oriented boundary: $\partial P_0 = \partial P_1 = \partial P_2 = \alpha$ (Fig. 2). Put $\mathbb{Y} = P_0 \cup P_1 \cup P_2$ and call this union a *book with three pages*. An embedding of a singular knot K in book $\mathbb{Y}$ is called a *3-page embedding*, if the following conditions hold:

1) all singular points of K lie on the axis α;
2) *finiteness*: the intersection $K \cap \alpha = A_1 \cup \cdots \cup A_m$ is a finite point set;
3) at every non-singular point $A_j \in K \cap \alpha$ two arcs lie in different half-planes;
4) a neighborhood of a singular point A_j lies in the plane $P_{i-1} \cup P_{i+1}$ for some $i \in \mathbb{Z}_3$;
5) *monotonicity*: for each $i \in \mathbb{Z}_3$ the restriction of the orthogonal projection $\mathbb{R}^3 \to \alpha \approx \mathbb{R}$ to each connected component of $K \cap P_i$ is a monotone function.

2.2. **Construction of a 3-page embedding from a plane diagram.** Let D be a plane diagram of a singular knot K, i.e. a planar 4-valent graph with vertices of two types: one corresponds to singular points of K and the other denotes the usual *crossings* in a planar representation of K. Given singular point B, let us mark two arcs with endpoint B, namely a *singular bridge* L_B lying on different branches of our singular knot. Also given crossing of the diagram D, mark a small segment (a *regular bridge*) in the overcrossing arc (Fig. 2).

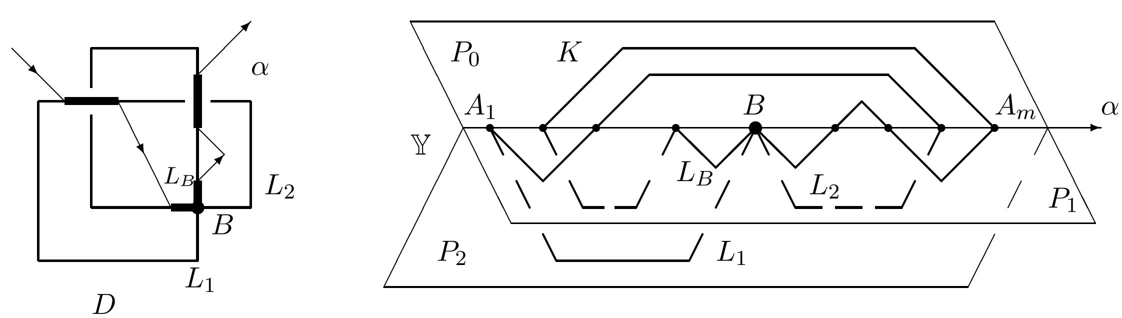

Fig. 2: Three-page embedding $K \subset \mathbb{Y}$, $w_K = a_0a_1b_2b_0x_0b_2d_2c_1c_2$.

Then take a non-self-intersected oriented path α in the plane of the diagram D with the following properties:

1) the endpoints of the path α lie far from D;
2) the path α traverses each bridge only once;

3) *transversality*: the path α intersects the diagram of D transversally beyond the bridges;

4) *balance*: for a singular point B consider two non-marked arcs L_1, L_2 with endpoint B, not containing the singular bridge L_B; then one of these arcs has to meet by the second endpoint the path α to the left of L_B and the other has to meet by the second endpoint the path α to the right of L_B (Fig. 2).

Such a path α can be easily found as follows: consider only bridges in the plane, i.e. finitely many arcs. Pass an arbitrary path α through each bridge satisfying 1) and 2). Then the transversality property 3) will hold, if we move our path α in general position with respect to the diagram D. Suppose that for the resulting path the balance property 4) does not hold for a singular point B, i.e. both non-marked arcs L_1, L_2 with endpoint B meet by the second endpoint the path α to the left of the bridge L_B (for example). Then slightly move the path α to the right of L_B using a move like the Reidemeister move $R2$ such that one of the two non-marked arcs L_1, L_2 (this is the arc L_2 in Fig. 2) meets by the second endpoint the path α to the right of L_B.

Now deform the plane of D in such a way that α becomes a straight line and the following *monotonicity* condition holds : the restriction of the orthogonal projection $\mathbb{R}^2 \to \alpha \approx \mathbb{R}$ to each connected component of $D - \alpha$ is a monotonic function. Denote by P_0 the upper half-plane over α and the lower half-plane by P_2 (Fig. 2). Finally, attach the third half-plane P_1 at α (at the reader's side) and push out all bridges into P_1 according to the following rules. Each regular bridge becomes a trivial arc. Each singular bridge becomes a broken line, which looks like the letter "W" that meets the axis α in its 3 upper vertices, and the middle one is the singular point B (Fig. 2). In fact, a neighborhood of any singular point can be embedded into the plane $P_0 \cup P_2$ not pushing out marked arcs into the third half-plane P_1. We used the notion of singular bridge for simplifying our argument.

2.3. **Encoding of three-page embeddings.** Each 3-page embedding is uniquely determined by its small neighborhood near the axis α in the book $\mathbb{Y}$. Indeed, in order to reconstruct the whole embedding it is sufficient to connect the opposite-directed arcs in each half-plane starting from interior arcs. We always mean that the half-plane P_0 is above the axis α, and the half-planes P_1, P_2 are below α. Moreover, we suppose that P_1 is above P_2, i.e. arcs in P_2 are drawn by dashed lines. Only the following 15 pictures may occur in a 3-page embedding of a singular knot near the axis α:

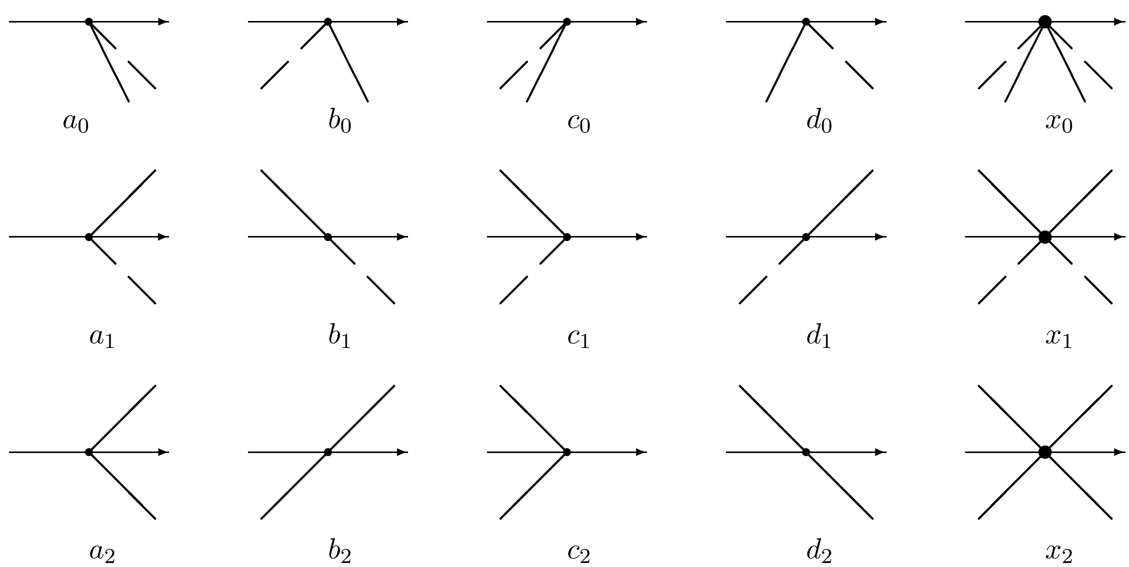

Fig. 3: Geometric interpretation of letters of the alphabet $\mathbb{A}$

Let W be the set of all words on the alphabet $\mathbb{A} = \{a_i, b_i, c_i, d_i, x_i \mid i \in \mathbb{Z}_3\}$ including the empty word $\varnothing$. For a given 3-page embedding of the knot K write one by one letters of $\mathbb{A}$ corresponding to the intersection points of $K \cap \alpha$. We obtain some word $w_K \in W$ (Fig. 2).

2.4. **Balanced words.** Note that by encoding of Subsection 2.3 one cannot obtain all the words of W. We call a word *balanced*, if it encodes some 3-page embedding. The following simple geometric criterion for a word to be balanced is available : in each half-plane P_i all arcs are connected with each other. Arcs of non-balanced 3-page embedding can go to infinity not meeting each other. One can easily rewrite this criterion algebraically in terms of the alphabet $\mathbb{A}$. For $i \in \mathbb{Z}_3$ a word w is called i*-balanced*, if after the following substitution

$$a_i, b_i, c_i, d_i, x_i \to \varnothing, \quad a_{i\pm1}, b_{i-1}, d_{i+1} \to (, \quad b_{i+1}, c_{i\pm1}, d_{i-1} \to), \quad x_{i\pm1} \to)($$

we obtain an expression with *completely balanced brackets* (or with *correctly placed brackets* in another terminology). This means that in each place the number of left brackets is not less than the number of the right ones, and their total numbers are equal. By W_i we denote the set of all i-balanced words in $\mathbb{A}$. Then a word w is called *balanced*, if it is i-balanced for each $i \in \mathbb{Z}_3$. So, the set of all balanced words is $W_b = W_0 \cap W_1 \cap W_2 \subset W$.

2.5. **Proof of Theorem 1.** Take a plane diagram D of a given singular knot K. Starting with the diagram D construct a 3-page embedding $K \subset \mathbb{Y}$ described in Subsection 2.2. Encode the obtained 3-page embedding of K by the balanced word $w_K \in W_b$ according to the rules of Subsection 2.3. Finally, consider the word w_K as an element of the semigroup SK. $\square$

3. Singular tangles

3.1. **Semigroup ST of singular tangles.** In order to prove Theorem 2 we need the notion of singular tangle. The category of tangles (without singularities) was studied by V. G. Turaev [23]. Take two horizontal semilines $\mathbb{R}_+ \subset \mathbb{R}^3$, for example given by coordinates: $(r, 0, 0)$ and $(r, 0, 1)$, where $r \in \mathbb{R}_+$. Mark the natural points $(j, 0, 0)$, $(k, 0, 1)$ for all $j, k \in \mathbb{N}$ on both semilines. Let Γ be a non-connected non-oriented infinite graph Γ with vertices of valency 1 and 4. A *singular tangle* is an embedding of Γ into the 3-dimensional layer $\{0 \le z \le 1\}$ such that (Fig. 4):

1) the set of the 1-valent vertices of the graph Γ coincides with the set of marked points

$$\{\ (j, 0, 0),\ (k, 0, 1) \mid j, k \in \mathbb{N}\ \};$$

2) all connected components of the graph Γ lying sufficiently far from the origin are the line segments connecting between the points $(k, 0, 0)$ and $(j, 0, 1)$ such that the difference $k - j$ is constant for all large j;

3) there exists a neighborhood of each 4-valent vertex of the graph Γ which lies in a plane.

We consider singular tangles up to ambient isotopy in the layer $\{0 \le z \le 1\}$ fixed on its boundary and such that condition 3) holds. Singular tangles can be represented by their plane diagrams analogous to singular knots (Fig. 4). One can obtain a *product* of singular tangles $\Gamma_1 \times \Gamma_2$ by attaching the top semiline of Γ_2 to the bottom semiline of Γ_1. So, the isotopy classes of singular tangles form some semigroup ST. The *unit* of ST is the singular tangle consisting of vertical segments. Let us introduce the singular tangles: ξ_k, η_k, σ_k, σ_k^{-1}, τ_k $(k \in \mathbb{N})$:

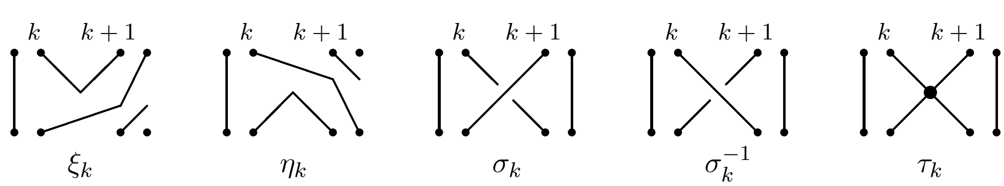

Fig. 4: Generators of the singular tangles

The following lemma transfers results of [23] from the classical case to ours.

Lemma 1. *The semigroup ST of singular tangles is generated by the elements ξ_k, η_k, σ_k, σ_k^{-1}, τ_k, $k \in \mathbb{N}$ (Fig. 4) and relations (11)-(23), where $k, l \in \mathbb{N}$:*

$$
\begin{array}{llllll}
(11) & \xi_k\xi_l = \xi_{l+2}\xi_k, & \xi_k\eta_l = \eta_{l+2}\xi_k, & \xi_k\sigma_l = \sigma_{l+2}\xi_k, & \xi_k\tau_l = \tau_{l+2}\xi_k & (l \geq k); \\
(12) & \eta_k\xi_l = \xi_{l-2}\eta_k, & \eta_k\eta_l = \eta_{l-2}\eta_k, & \eta_k\sigma_l = \sigma_{l-2}\eta_k, & \eta_k\tau_l = \tau_{l-2}\eta_k & (l \geq k+2); \\
(13) & \sigma_k\xi_l = \xi_l\sigma_k, & \sigma_k\eta_l = \eta_l\sigma_k, & \sigma_k\sigma_l = \sigma_l\sigma_k, & \sigma_k\tau_l = \tau_l\sigma_k & (l \geq k+2); \\
(14) & \tau_k\xi_l = \xi_l\sigma_k, & \tau_k\eta_l = \eta_l\sigma_k, & \tau_k\sigma_l = \sigma_l\sigma_k, & \tau_k\tau_l = \tau_l\tau_k & (l \geq k+2);
\end{array}
$$

$$
\begin{array}{llll}
(15) & \eta_{k+1}\xi_k = 1 = \eta_k\xi_{k+1}; & (16) & \eta_{k+2}\sigma_{k+1}\xi_k = \sigma_k^{-1} = \eta_k\sigma_{k+1}\xi_{k+2}; \\
(17) & \eta_{k+2}\tau_{k+1}\xi_k = \tau_k = \eta_k\tau_{k+1}\xi_{k+2}; & (18) & \eta_k\sigma_k = \eta_k,\ \sigma_k\xi_k = \xi_k; \\
(19) & \sigma_k\sigma_k^{-1} = 1 = \sigma_k^{-1}\sigma_k; & (20) & \sigma_k\sigma_{k+1}\sigma_k = \sigma_{k+1}\sigma_k\sigma_{k+1}; \\
(21) & \sigma_k\sigma_{k+1}\tau_k = \tau_{k+1}\sigma_k\sigma_{k+1}; & (22) & \tau_k\sigma_{k+1}\sigma_k = \sigma_{k+1}\sigma_k\tau_{k+1}; \\
(23) & \sigma_k\tau_k = \tau_k\sigma_k. & &
\end{array}
$$

Proof. Recall that we work in the PL-category. It means that a given singular tangle in the layer $\{0 \leq z \leq 1\}$ consists of finite broken lines. The local maxima and minima of the height function are called *extremal points*. We call a *peculiarity* of a diagram of a tangle either a 4-valent vertex, or a crossing, or an extremal point. We say that a singular tangle is in *general position* if its plane diagram satisfies the following conditions:

1) the set of all peculiarities is finite;
2) crossings do not coincide with extremal points;
3) for each 4-valent vertex two arcs go up, and the rest two go down;
4) each horizontal line (that is parallel to the Ox-axis) contains at most than one peculiarity.

Obviously, by a slight deformation every tangle can be moved in a general position. Then the tangle diagram is splitted by horizontal lines into bands such that each of them contains only one peculiarity. Considering the peculiarities from the top to the bottom one by one, write the corresponding generators from Fig. 4 from left to right. Namely, the generators ξ_k, η_k represent extremal points; the generators σ_k, σ_k^{-1} correspond to crossings; τ_k presents a 4-valent vertex. It suffices to show that every ambient isotopy of singular tangles decomposes into "elementary isotopies" corresponding to relations (11)-(23). It follows from the Reidemeister theorem [16] and the general position arguments that an arbitrary isotopy of singular tangles can be decomposed into the following moves:

1) general position isotopy;
2) swopping of heights of two peculiarities;
3) creation or annihilation of a couple of close extremal points;
4) an isotopy of a crossing or a 4-valent vertex near extremal point;
5) the Reidemeister moves $R1 - R5$ (Fig. 1).

The first type isotopies keep the constructed word in the letters ξ_k ,η_k, σ_k, σ_k^{-1}, τ_k, $k \in \mathbb{N}$. The second type isotopies are desribed by relations (11)-(14); the third type isotopies correspond to relations (15). In [23, proof of lemma 3.4] it was shown that in the smooth category all isotopies of a crossing near extremal point are geometrically decomposed into relations (16). Similarly, in our PL-case we can check that relations (17) are sufficient to isotope a 4-valent vertex near extremal point. Finally, Reidemeister moves $R1 - R5$ correspond to relations (18)-(23), respectively. □

3.2. **Three-page singular tangles.** A notion of 3-page singular tangle will be used in the proofs of Theorems 2 and 3. Consider three semi-lines in the horizontal plane $\{z = 0\}$ having a common endpoint. Let it be for example:

$$T = \{x \geq 0, y = z = 0\} \cup \{y \geq 0, x = z = 0\} \cup \{x \leq 0, y = z = 0\} \subset \{z = 0\}.$$

Mark the integer points on the semilines: $\{(j, 0, 0), (0, k, 0), (-l, 0, 0) \mid j, k, l \in \mathbb{N}\}$. Let I be an interval connecting the points $(0, 0, 0)$ and $(0, 0, 1)$. Put:

$$P_0 = \{x \geq 0, y = z = 0\} \times I, \quad P_1 = \{y \geq 0, x = z = 0\} \times I, \quad P_2 = \{x \leq 0, y = z = 0\} \times I.$$

Formally, here P_i is not a half-plane, but a band $I \times \mathbb{R}$, which we call a *page*. The book $\mathbb{Y}$ of Subsection 2.1 is the interior of the set $T \times I$, i.e. in Section 2 we considered the embeddings $K \subset T \times I$

such that $K \cap \partial(T \times I) = \emptyset$. Let Γ be a non-connected non-oriented infinite graph with vertices of valency 1 and 4. A *three-page singular tangle* is an embedding of Γ into a book $T \times I$ such that (Fig. 5):

1) the set of 1-valent vertices of the graph Γ coincides with the set of the marked points

$$\{(j,0,0),(j,0,1),(0,k,0),(0,k,1),(-l,0,0),(-l,0,1),\ \mid j,k,l \in \mathbb{N}\};$$

2) all 4-valent vertices of Γ lie in the segment I;
3) *finiteness*: the intersection $\Gamma \cap I = A_1 \cup \dots \cup A_m$ is a finite point set;
4) the two arcs of any 2-valent vertex $A_j \in \Gamma \cap I$ lie in different half-planes;
5) a neighborhood of each 4-valent vertex of Γ lies in one pair of pages from P_0, P_1, P_2;
6) *monotonicity*: for every $i \in \mathbb{Z}_3$ restriction of the orthogonal projection $T \times I \to I \approx [0,1]$ to each connected component of $\Gamma \cap P_i$ is a monotone function.
7) for each $i \in \mathbb{Z}_3$ all connected components of the graph Γ lying in a plane P_i sufficiently far from the origin are parallel line segments.

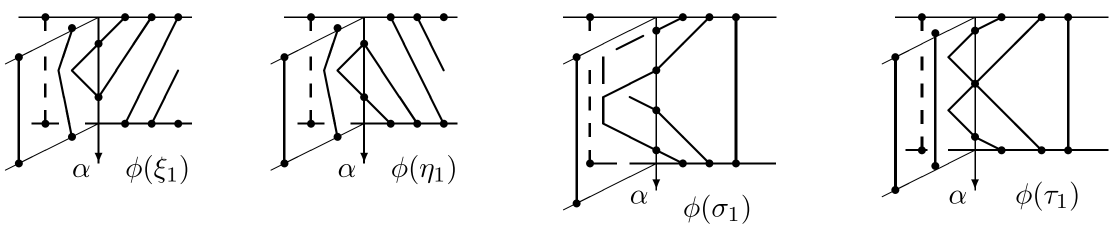

Fig. 5: Three-page singular tangles

As for singular tangles from Subsection 3.1, isotopy classes of three-page tangles in the layer $\{0 \le z \le 1\}$ form a semigroup. Each three-page tangle can be encoded by a word in the alphabet $\mathbb{A} = \{a_i, b_i, c_i, d_i, x_i \mid i \in \mathbb{Z}_3\}$ (Fig. 3) in the same way as in Subsection 2.3. A three-page tangle is called *almost balanced*, if the corresponding word in the alphabet $\mathbb{A}$ is 1-balanced and 2-balanced (see Subsection 2.4). Note that for any i-balanced 3-page tangle all strings in the band P_i can be assumed vertical. By BT we denote the semigroup of almost balanced 3-page tangles. Define the map $\varphi : ST \to BT$ on the generators as follows, $k \in \mathbb{N}$ (Fig. 5):

(24) $\phi(\xi_k) = d_2^k c_2 b_2^{k-1}, \phi(\eta_k) = d_2^{k-1} a_2 b_2^k, \phi(\sigma_k) = d_2^{k-1} b_1 d_2 d_1 b_2^k, \phi(\sigma_k^{-1}) = d_2^k b_1 b_2 d_1 b_2^{k-1}, \phi(\tau_k) = d_2^k x_2 b_2^k.$

Each tangle goes to the corresponding three-page embedding plus vertical intervals. The following Lemma is proved in the same way as [12, Lemma 3].

Lemma 2. *The map $\varphi : ST \to BT$ is well-defined isomorphism of semigroups.*

Proof. First let us check that isotopy equivalent singular tangles go to isotopy equivalent 3-page singular tangles under the map φ. Actually, by definition, singular tangles from semigroups ST and BT are considered up to isotopy in the layer $\{0 < z < 1\}$. So the injectivity of the map φ follows. Now we construct the inverse map $\psi : BT \to ST$. Let us associate with each almost balanced 3-page tangle $\Gamma \in BT$ the singular tangle $\psi(\Gamma) \in ST$ given by the following diagram. According to the almost balance we consider that all segments of Γ lying in the pages P_1, P_2 are vertical. Deleting all these vertical segments from Γ we obtain some graph-tangle $\psi(\Gamma)$ in the sense of Subsection 3.1. Clearly, the composition $\psi \circ \varphi : ST \to ST$ is identical on the generators (Fig. 5). So, the maps φ, ψ are mutually inverse. $\square$

3.3. **Classification of three-page singular tangles.** By $\varphi(11) - \varphi(23)$ we denote the relations between words in the alphabet $\mathbb{A}$ which are obtained from the relations (11)-(23) of the semigroup ST under the isomorphism φ (Lemma 2). The following Lemma will be proved in Subsection 4.3.

Lemma 3. *Relations (1)-(10) follow from relations $\varphi(11) - \varphi(23)$ of the semigroup SK.*

Theorem 2 is a special case of the following classification theorem for three-page singular tangles, which we prove by analogy with Theorem 1 of [12].

Theorem 4. *The semigroup of the isotopy classes of 3-page singular tangles is isomorphic to the semigroup SK.*

Proof. As it was already mentioned in Subsection 3.2 with each three-page singular tangle it is possible to associate a word in the alphabet $\mathbb{A}$ and hence an element of the semigroup SK. Conversely, each element of the semigroup SK can be completed to some three-page tangle, if we add three families of parallel segments on each page P_i, $i \in \mathbb{Z}_3$. For example, the three-page tangles corresponding to the following elements from SK are depicted on Fig. 3: d_2c_2, a_2b_2, $b_1d_2d_1b_2$, $d_2x_2b_2$. Relations (1)-(10) of the semigroup SK can be easily performed by an ambient isotopy in the layer $\{0 < z < 1\}$. Hence it remains to prove that each isotopy of three-page singular tangle can be decomposed into "elementary isotopies" corresponding to relations (1)-(10) of SK. It suffices to do this for almost balanced three-page tangles. Really, for a 3-page tangle given by a word w let n_1 and m_1 (respectively, n_2 and m_2) be the maximal numbers of points on the semilines $P_1 \cap \{z = 0\}$ and $P_1 \cap \{z = 1\}$ (respectively, on the semilines $P_2 \cap \{z = 0\}$ and $P_2 \cap \{z = 1\}$) that are connected by arcs with points on the segment I. For letter a_0 one gets $n_1 = n_2 = 0, m_1 = m_2 = 1$ (Fig. 2a). For an almost balanced 3-page graph-tangle we have $n_1 = m_1 = n_2 = m_2 = 0$. Then the word $b_1^{n_2} d_2^{n_1} w b_2^{m_1} d_1^{m_2}$ is almost balanced. Because of invertibility of generators b_i, d_i such a transformation sends equivalent words to equivalent. By Lemma 2 we can associate a singular tangle in the sense of Subsection 3.1 with each almost balanced word. For such tangles each isotopy is already decomposed into the elementary isotopies corresponding to relations $\varphi(11) - \varphi(23)$ (Lemmas 1 and 2). So Lemma 3 finishes the proof of Theorem 4. □

3.4. **Proof of Theorem 3.** We will identify an arbitrary three-page singular tangle with the corresponding element of the semigroup SK. A 3-page tangle is called *knot-like* if it contains a singular knot near the axis α and the rest of it consists only of vertical segments. Evidently, the knot-like tangles correspond to balanced words from $W_b \subset W$.

Lemma 4. *An element $w \in SK$ defines a knot-like tangle if and only if w is a central element in the semigroup SK.*

Proof. The part "only if" is geometrically evident: a singular knot can be moved by an isotopy to any place of a given tangle, i.e., a knot-like element commutes with any other by Theorem 2. Let w be a central element in SK. Then for each $k \in \mathbb{N}$ we have $b_i^k d_i^k w = w b_i^k d_i^k$. Denote by m (respectively by n) the number of arcs of the 3-page tangle w that go out in the page P_{i-1} to the left (respectively, to the right) boundary. Then for sufficiently large k the number of arcs of the 3-page tangle $b_i^k d_i^k w$ that go out in the page P_{i-1} to the left boundary is equal to k, and for the tangle $w b_i^k d_i^k$ is equal to $m + k - n$, i.e. $m = n$. Hence for sufficiently large l and any $j = i - 1 \in \mathbb{Z}_3$ the word $a_0^l a_1^l w c_1^l c_0^l$ is j-balanced, so it is balanced. Since w is central, the word $w a_0^l a_1^l c_1^l c_0^l$ is also balanced. Then it is geometrically obvious that the element w defines a knot-like tangle. □

Theorem 3 follows from Lemma 4.

3.5. **Classification of knuckle 4-valent graphs.** The isotopy classification problem of such graphs was considered in the paper [17]. Dynnikov's method gives a possibility of solving it analogously to the case of singular knots. Let us introduce a semigroup FG with the same generators and relations as the semigroup SK, only we change relation (6) to the following: $(6') \quad x_i(d_{i+1}d_i d_{i-1}) = x_i$. Then the semigroup FG has 15 generators and 84 defining relations.

Theorem 5. *The center of the semigroup FG classifies all non-oriented knuckle 4-valent graphs in $\mathbb{R}^3$ up to an ambient isotopy.*

Theorem 5 is proved analogously to Theorems 1–3 with the change of relation (6) to (6′) and relation (23) in Lemma 1 to the following (23′) $\sigma_k \tau_k = \tau_k$.

4. Proof of Propositions 1 and 2

In Claim 1 we obtain new word equivalences from relations (1)-(10) of the semigroup SK. In Subsection 4.2, using Claims 1–3 we prove Lemma 5 about decomposition of any i-balanced word. Lemma 5 and Claim 6 reduce the infinite number of relations $\varphi(11) - \varphi(23)$ to the finite number of relations (1)-(10). The proof of Lemma 3 finishes in Subsection 4.3 using Claims 5 and 6. All relations will be obtained in a formal way, but they have a geometric interpretation (Fig. 3).

4.1. Corollaries of relations (1)-(10).

Claim 1. *Equivalences (1)-(10) imply the following (we suppose that $i \in \mathbb{Z}_3$ and $w_i \in \mathbb{B}_i = \{a_i,\ b_i,\ c_i,\ d_i,\ b_{i-1}b_id_{i-1},\ b_{i-1}d_id_{i-1}\ \}$):*

(25) $b_i \sim d_{i+1}d_{i-1}$, *or* $b_{i+1} \sim d_{i-1}d_i,\ b_{i-1} \sim d_id_{i+1}$, *or* (25) $b_0 \sim d_1d_2,\ b_1 \sim d_2d_0,\ b_2 \sim d_0d_1$;
(26) $d_i \sim b_{i-1}b_{i+1}$, *or* $d_{i-1} \sim b_{i+1}b_i,\ d_{i+1} \sim b_ib_{i-1}$, *or* (26) $d_0 \sim b_2b_1,\ d_1 \sim b_0b_2,\ d_2 \sim b_1b_0$;
(27) $d_{i+1}b_{i-1} \sim b_{i-1}d_{i+1}t_i,\ b_{i+1}d_{i-1} \sim t_id_{i-1}b_{i+1}$, *where* $t_i = b_{i+1}d_{i-1}d_{i+1}b_{i-1}$;

(28) $a_i \sim a_{i-1}b_{i+1},\ c_i \sim d_{i+1}c_{i-1}$; (29) $a_ib_i \sim a_{i-1}d_{i-1},\ d_ic_i \sim b_{i-1}c_{i-1}$;
(30) $b_i \sim a_ib_ic_i,\ d_i \sim a_id_ic_i$;
(31) $b_{i-1}x_{i+1}d_{i-1} \sim x_i$; (32) $b_ix_id_i \sim d_{i+1}x_{i+1}b_{i+1}$;
(33) $(d_ic_i)w_{i+1} \sim w_{i+1}(d_ic_i)$; (34) $(b_ic_i)w_{i-1} \sim w_{i-1}(b_ic_i)$;
(35) $(a_ib_i)w_{i+1} \sim w_{i+1}(a_ib_i)$; (36) $(a_id_i)w_{i-1} \sim w_{i-1}(a_id_i)$;
(37) $t_iw_i \sim w_it_i,\ t'_iw_i \sim w_it'_i$, *where* $t_i = b_{i+1}d_{i-1}d_{i+1}b_{i-1},\ t'_i = d_{i-1}b_{i+1}b_{i-1}d_{i+1}$;
(38) $(d_ix_ib_i)w_{i+1} \sim w_{i+1}(d_ix_ib_i)$; (39) $(b_ix_id_i)w_{i-1} \sim w_{i-1}(b_ix_id_i)$;
(40) $d_{i+1}b_{i-1}w_id_{i-1}b_{i+1} \sim b_{i-1}d_{i+1}w_ib_{i+1}d_{i-1}$;
(41) $b^2_{i-1}a_id^2_{i-1} \sim (b_{i-1}a_id_{i-1})d^2_i(b_{i-1}b_id_{i-1})b_i$; (42) $b^2_{i-1}c_id^2_{i-1} \sim d_i(b_{i-1}d_id_{i-1})b^2_i(b_{i-1}c_id_{i-1})$;
(43) $b^2_{i-1}b_id^2_{i-1} \sim (b_{i-1}b_id_{i-1})d^2_i(b_{i-1}b_id_{i-1})b_i$; (44) $b^2_{i-1}d_id^2_{i-1} \sim d_i(b_{i-1}d_id_{i-1})b^2_i(b_{i-1}d_id_{i-1})$;

(45) $b^2_{i-1}x_id^2_{i-1} \sim (b^2_{i-1}b_id^2_{i-1})(b_{i-1}d_id_{i-1})d^2_ix_ib^2_i(b_{i-1}b_id_{i-1})(b^2_{i-1}d_id^2_{i-1})$.

Proof. Note that equivalences (25)-(27) follow easily from (3)-(4). By (4) we have $d_i \sim b_i^{-1}$, $b_{i-1}b_id_{i-1} \sim (b_{i-1}d_id_{i-1})^{-1}$, and $t'_i \sim t_i^{-1}$. Then (35), (37) and (38) follow from (8), (9) and (10) respectively. The other equivalences will be verified step by step, using those already checked. Recall that $i \in \mathbb{Z}_3 = \{0,1,2\}$, i.e. in $\mathbb{Z}_3$ we have $(i+1)+1 = i-1$ and $(i-1)-1 = i-1$.

(28) $a_{i-1}b_{i+1} \overset{(1)}{\sim} (a_id_{i+1})b_{i+1} \overset{(4)}{\sim} a_i$, $\quad d_{i+1}c_{i-1} \overset{(1)}{\sim} d_{i+1}(b_{i+1}c_i) \overset{(4)}{\sim} c_i$;
(29) $a_ib_i \overset{(28)}{\sim} (a_{i-1}b_{i+1})b_i \overset{(26)}{\sim} a_{i-1}d_{i-1}$, $\quad d_ic_i \overset{(26)}{\sim} (b_{i-1}b_{i+1})c_i \overset{(1)}{\sim} b_{i-1}c_{i-1}$;
(30) $a_ib_ic_i \overset{(29)}{\sim} a_i(d_{i+1}c_{i+1}) \overset{(1)}{\sim} a_{i-1}c_{i+1} \overset{(1)}{\sim} b_i$, $\quad a_id_ic_i \overset{(29)}{\sim} a_i(b_{i-1}c_{i-1}) \overset{(28)}{\sim} a_{i+1}c_{i-1} \overset{(1)}{\sim} d_i$;
(31) $b_{i-1}x_{i+1}d_{i-1} \overset{(2)}{\sim} b_{i-1}(d_{i-1}x_ib_{i-1})d_{i-1} \overset{(4)}{\sim} x_i$;
(32) $b_ix_id_i \overset{(31)}{\sim} b_i(b_{i-1}x_{i+1}d_{i-1})d_i \overset{(25),(26)}{\sim} d_{i+1}x_{i+1}b_{i+1}$.

Below in the proof of (33) we firstly commute b_{i+1} with d_ic_i, and then we use this relation to commute a_{i+1} with d_ic_i.

(33b) $b_{i+1}(d_ic_i) \overset{(30)}{\sim} (a_{i+1}b_{i+1}c_{i+1})(d_ic_i) \overset{(7)}{\sim} a_{i+1}b_{i+1}(d_ic_i)c_{i+1} \overset{(26)}{\sim} a_{i+1}b_{i+1}(b_{i-1}b_{i+1})c_ic_{i+1} \overset{(1)}{\sim}$
$\overset{(1)}{\sim} (a_{i+1}b_{i+1})b_{i-1}c_{i-1}c_{i+1} \overset{(8)}{\sim} b_{i-1}c_{i-1}(a_{i+1}b_{i+1})c_{i+1} \overset{(30)}{\sim} b_{i-1}c_{i-1}b_{i+1} \overset{(1)}{\sim} b_{i-1}(b_{i+1}c_i)b_{i+1} \overset{(26)}{\sim} (d_ic_i)b_{i+1}$;

(33a) $a_{i+1}(d_ic_i) \overset{(1)}{\sim} (a_{i-1}d_i)(d_ic_i) \overset{(26)}{\sim} a_{i-1}(b_{i-1}b_{i+1})(d_ic_i) \overset{(33b)}{\sim} a_{i-1}b_{i-1}(d_ic_i)b_{i+1} \overset{(35)}{\sim}$
$\overset{(35)}{\sim} (d_ic_i)(a_{i-1}b_{i-1})b_{i+1} \overset{(26)}{\sim} (d_ic_i)(a_{i-1}d_i) \overset{(1)}{\sim} (d_ic_i)a_{i+1}$.

The other equivalences in (33) follow from (33a), (33b) and (7). Equivalences (34), (36),(39) follow respectively from (29) and (33), (29) and (35), (32) and (38). The last calculations are trivial:

(40) $d_{i+1}b_{i-1}w_id_{i-1}b_{i+1} \overset{(27)}{\sim} (b_{i-1}d_{i+1}t_i)w_id_{i-1}b_{i+1} \overset{(39)}{\sim} b_{i-1}d_{i+1}(w_it_i)d_{i-1}b_{i+1} \overset{(27)}{\sim} b_{i-1}d_{i+1}w_ib_{i+1}d_{i-1}$;

(41) $b^2_{i-1}a_id^2_{i-1} \overset{(4)}{\sim} b^2_{i-1}a_i(d_ib_i)d^2_{i-1} \overset{(36)}{\sim} b_{i-1}(a_id_i)(b_{i-1}b_i)d^2_{i-1} \overset{(26)}{\sim} b_{i-1}a_id_i(b_{i-1}b_i)d_{i-1}(b_{i+1}b_i) \overset{(37)}{\sim}$
$\overset{(37)}{\sim} b_{i-1}a_ib_{i+1}(d_ib_{i-1}b_id_{i-1})b_i \overset{(25)}{\sim} (b_{i-1}a_id_{i-1})d^2_i(b_{i-1}b_id_{i-1})b_i$;

(42) $b^2_{i-1}b_id^2_{i-1} \overset{(4)}{\sim} b_{i-1}(b_id_i)b_{i-1}b_id^2_{i-1} \overset{(26)}{\sim} b_{i-1}b_i(d_ib_{i-1}b_id_{i-1})(b_{i+1}b_i) \overset{(37)}{\sim}$
$\overset{(37)}{\sim} b_{i-1}b_ib_{i+1}(d_ib_{i-1}b_id_{i-1})b_i \overset{(25)}{\sim} (b_{i-1}b_id_{i-1})d^2_i(b_{i-1}b_id_{i-1})b_i$;

(43) $b_{i-1}^2 c_i d_{i-1}^2 \overset{(4)}{\sim} b_{i-1}^2 (d_i b_i) c_i d_{i-1}^2 \overset{(34)}{\sim} b_{i-1}^2 d_i d_{i-1} (b_i c_i) d_{i-1} \overset{(25)}{\sim} (d_i d_{i+1})(b_{i-1} d_i d_{i-1} b_i) c_i d_{i-1} \overset{(37)}{\sim}$
$\overset{(37)}{\sim} d_i (b_{i-1} d_i d_{i-1} b_i) d_{i+1} c_i d_{i-1} \overset{(26)}{\sim} d_i (b_{i-1} d_i d_{i-1}) b_i^2 (b_{i-1} c_i d_{i-1})$;

(44) $b_{i-1}^2 d_i d_{i-1}^2 \overset{(4)}{\sim} b_{i-1}^2 d_i d_{i-1} (b_i d_i) d_{i-1} \overset{(25)}{\sim} (d_i d_{i+1}) b_{i-1} d_i d_{i-1} b_i d_i d_{i-1} \overset{(37)}{\sim}$
$\overset{(37)}{\sim} d_i (b_{i-1} d_i d_{i-1} b_i) d_{i+1} d_i d_{i-1} \overset{(26)}{\sim} d_i (b_{i-1} d_i d_{i-1}) b_i^2 (b_{i-1} d_i d_{i-1})$;

(45) $b_{i-1}^2 x_i d_{i-1}^2 \overset{(4)}{\sim} b_{i-1}^2 (b_i d_i) x_i (b_i b_{i+1}^2 d_{i+1}^2 d_i) d_{i-1}^2 \overset{(10)}{\sim} b_{i-1}^2 b_i b_{i+1}^2 (d_i x_i b_i) d_{i+1}^2 d_i d_{i-1}^2 \overset{(25),(26)}{\sim}$
$\overset{(25),(26)}{\sim} b_{i-1}^2 b_i (d_{i-1} d_i)^2 d_i x_i b_i (b_i b_{i-1})^2 d_i d_{i-1}^2 \overset{(4)}{\sim} (b_{i-1}^2 b_i d_{i-1}^2)(b_{i-1} d_i d_{i-1}) d_i^2 x_i b_i^2 (b_{i-1} b_i d_{i-1})(b_{i-1}^2 d_i d_{i-1}^2)$.

□

4.2. **Decomposition of i-balanced words.**

Claim 2. *For each $i \in \mathbb{Z}_3$ any i-balanced word is equivalent to some i-balanced word that contains only the following letters: a_i, b_i, c_i, d_i, x_i, b_{i-1}, d_{i-1}.*

Proof. Using the following substitutions, we can eliminate the other letters:

$$\begin{cases} a_{i-1} \overset{(1)}{\sim} a_i d_{i+1}, \quad a_{i+1} \overset{(28)}{\sim} a_i b_{i-1}, \quad c_{i-1} \overset{(1)}{\sim} b_{i+1} c_i, \quad c_{i+1} \overset{(28)}{\sim} d_{i-1} c_i, \\ b_{i+1} \overset{(25)}{\sim} d_{i-1} d_i, \quad d_{i+1} \overset{(26)}{\sim} b_i b_{i-1}, \quad x_{i-1} \overset{(2)}{\sim} d_i x_{i+1} b_i, \quad x_{i+1} \overset{(2)}{\sim} d_{i-1} x_i b_{i-1}. \end{cases}$$

□

Fix an index $i \in \mathbb{Z}_3$. Let w be an i-balanced word on the letters a_i, b_i, c_i, d_i, x_i, b_{i-1}, d_{i-1}. Consider the substitution $\mu : a_i, b_i, c_i, d_i, x_{m,i} \to \bullet$, $b_{i-1} \to ($, $d_{i-1} \to)$. Denote by $\mu(w)$ the resulting *encoding* consisting of brackets and bullets. Because the given word w is i-balanced, so the encoding $\mu(w)$ (without bullets) is a balanced bracket expression. For each place k denote by $dif(k)$ the difference between the number of left and right brackets in a subword of $\mu(w)$, ending at this place. The maximum of $dif(k)$ for all k we call the *depth* of K: $d(w)$. For example, the word $w = b_{i-1}^2 a_i d_{i-1}^2$ has the encoding $\mu(w) = ((\bullet))$ and the depth $d(w) = 2$.

By the *star of depth* k we call an encoding of the type $(^k \bullet)^k$ which has k couples of brackets. The bullet is a star of depth 0. If for a word w its encoding $\mu(w)$ decomposes into several stars, then w is called *star decomposable*. In this case the depth $d(w)$ is maximal among the depths of all stars participating in the decomposition.

Claim 3. *Every i-balanced word w is equivalent to some star decomposable word w' of the same depth $d(w') = d(w)$.*

Proof. Consider the beginning of the encoding $\mu(w)$. After several initial left brackets $\mu(w)$ contains either a right bracket or a bullet. In the first case we delete a couple of brackets () by the rule: $b_{i-1} d_{i-1} \overset{(2)}{\sim} \emptyset$. Hence we can suppose that the next simbol after the sequence of k left brackets is a bullet. Because $\mu(w)$ is balanced, then after this bullet it may be the sequence of j, $0 \le j \le k$, right brackets. If $j < k$, then insert into w after the last right bracket the following subword $d_{i-1}^{k-j} b_{i-1}^{k-j} \overset{(2)}{\sim} \emptyset$. This operation does not change the depth $d(w)$. Then in the resulting word w_1 the encoding $\mu(w_1)$ contains a star of depth k at the beginning. Continuing this process, after a finite number of steps we get a star decomposable word w_N of the same depth $d(w_N) = d(w)$. □

For any letter s denote by s' the word $b_{i-1} s d_{i-1}$, for example, $a_i' = b_{i-1} a_i d_{i-1}$.

Claim 4. *Each star decomposable word w is equivalent to a word decomposed into the following i-balanced subwords: $\{a_i, b_i, c_i, d_i, x_i, a_i', b_i', c_i', d_i', x_i'\}$.*

Proof. We use induction on the depth $d(w)$. The case $d(w) = 1$ is trivial. Let $\mu(w)$ contain a star of depth $k \geq 2$. Apply one of the following transformations to every such star.

$$\begin{cases} u = b_{i-1}^2 a_i d_{i-1}^2 \overset{(46a)}{\sim} a_i' d_i^2 b_i' b_i = v, \text{ i.e. } \mu(u) = ((\bullet)) \to \mu(v) = (\bullet) \bullet \bullet (\bullet) \bullet; \\ u = b_{i-1}^2 b_i d_{i-1}^2 \overset{(46b)}{\sim} b_i' d_i^2 b_i' b_i = v, \text{ i.e. } \mu(u) = ((\bullet)) \to \mu(v) = (\bullet) \bullet \bullet (\bullet) \bullet; \\ u = b_{i-1}^2 c_i d_{i-1}^2 \overset{(46c)}{\sim} d_i d_i' b_i^2 c_i' = v, \text{ i.e. } \mu(u) = ((\bullet)) \to \mu(v) = \bullet (\bullet) \bullet \bullet (\bullet); \\ u = b_{i-1}^2 d_i d_{i-1}^2 \overset{(46c)}{\sim} d_i d_i' b_i^2 d_i' = v, \text{ i.e. } \mu(u) = ((\bullet)) \to \mu(v) = \bullet (\bullet) \bullet \bullet (\bullet); \end{cases}$$

$$\begin{cases} u = b_{i-1}^2 x_i d_{i-1}^2 \overset{(46x)}{\sim} (b_{i-1}^2 b_i d_{i-1}^2)(b_{i-1} d_i d_{i-1}) d_i^2 x_i b_i^2 (b_{i-1} b_i d_{i-1})(b_{i-1}^2 d_i d_{i-1}^2) \overset{(46b,46d)}{\sim} \\ \overset{(46b,46d)}{\sim} (b_i' d_i^2 b_i' b_i) d_i' d_i^2 x_i b_i^2 b_i' (d_i d_i' b_i^2 d_i') = v, \text{ i.e.} \\ \mu(u) = ((\bullet)) \to \mu(v) = (\bullet) \bullet \bullet (\bullet) \bullet (\bullet) \bullet \bullet \bullet \bullet \bullet (\bullet) \bullet (\bullet) \bullet \bullet (\bullet). \end{cases}$$

We get a word $w_1 \sim w$, equivalent to a star decomposed word w_2 of depth $d(w_2) = d(w) - 1$ according to Claim 3. This finishes the induction step. □

Lemma 5. *For all $i \in \mathbb{Z}_3$ each i-balanced word is equivalent to some word which can be decomposed into the i-balanced words of the set $\mathbb{B}_i = \{a_i, b_i, c_i, d_i, x_i, b_{i-1} b_i d_{i-1}, b_{i-1} d_i d_{i-1}\}$.*

Proof. By Claims 3 and 4 it remains to eliminate only the following words:

$$a_i' = b_{i-1} a_i d_{i-1} \overset{(25)}{\sim} (d_i d_{i+1}) a_i d_{i-1} \overset{(4)}{\sim} d_i d_{i+1} a_i (b_i d_i) d_{i-1} \overset{(35)}{\sim} d_i (a_i b_i) d_{i+1} d_i d_{i-1} \overset{(26)}{\sim} d_i a_i b_i^2 (b_{i-1} d_i d_{i-1});$$
$$c_i' = b_{i-1} c_i d_{i-1} \overset{(26)}{\sim} b_{i-1} c_i (b_{i+1} b_i) \overset{(4)}{\sim} b_{i-1} (b_i d_i c_i) b_{i+1} b_i \overset{(33)}{\sim} b_{i-1} b_i b_{i+1} (d_i c_i) b_i \overset{(35)}{\sim} (b_{i-1} b_i d_{i-1}) d_i^2 c_i b_i;$$

$$\begin{cases} x_i' = b_{i-1} x_i d_{i-1} \overset{(4)}{\sim} b_{i-1} (b_i b_{i+1} d_{i+1} d_i) x_i (b_i d_i) d_{i-1} \overset{(38)}{\sim} b_{i-1} b_i b_{i+1} (d_i x_i b_i) d_{i+1} d_i d_{i-1} \overset{(25)}{\sim} \\ \overset{(25)}{\sim} (b_{i-1} b_i d_{i-1}) d_i^2 x_i b_i d_{i+1} d_i d_{i-1} \overset{(26)}{\sim} (b_{i-1} b_i d_{i-1}) d_i^2 x_i b_i^2 (b_{i-1} d_i d_{i-1}). \end{cases}$$

□

Denote by $(33') - (40')$ relations (33)-(40), assuming that $w_i \in W_i$.

Claim 5. *Generalized equivalences $(33') - (40')$ hold for arbitrary i-balanced words $w_i \in W_i$.*

Proof. By Lemma 5 every i-balanced word of W_i can be decomposed into the elementary words from $\mathbb{B}_i$. Since equivalences (33)-(40) hold for words from $\mathbb{B}_i$, they also hold for words from W_i. Note that we get an infinite number of new equivalences $(33') - (40')$. □

4.3. **Deduction of relations $\varphi(11) - \varphi(23)$ from relations (1)-(10) of the semigroup SK.** For each $l \geq 1$ denote by u_l the symbol from the set of generators $\{\xi_l, \eta_l, \sigma_l, \sigma_l^{-1}, \tau_l\}$ of the semigroup ST. Define the *shift maps* $\theta_k : ST \to ST$ and $\rho_k : BT \to BT$ by $\theta_k(u_l) = u_{k+l}$ and $\rho_k(w) = d_2^k w b_2^k$. Evidently, the shift map $\theta_k : ST \to ST$ is a well-defined homomorphism. Really, each relation from (11)-(23) for $k > 1$ is obtained from the corresponding relation for $k = 1$ by the shift map θ_{k-1}. For example, relation $\xi_k \xi_l = \xi_{l+2} \xi_k$ is obtained from $\xi_1 \xi_{l-k+1} = \xi_{l-k+3} \xi_1$ by the shift map θ_{k-1}. By (2) the shift map ρ_k sends equivalent words to equivalent ones, i.e. ρ_k is a homomorphism. Moreover, the following diagram

$$\begin{array}{ccc} ST & \xrightarrow{\theta_k} & ST \\ {\scriptstyle\varphi}\downarrow & & \downarrow{\scriptstyle\varphi} \\ BT & \xrightarrow{\rho_k} & ST \end{array}$$

is commutative, that implies Claim 6.

Claim 6. *For each $k \in \mathbb{N}$ relations $\varphi(11) - \varphi(23)$ can be obtained from relations $\varphi(11) - \varphi(23)$ for $k = 1$ on using the equivalences $b_2 d_2 \sim 1 \sim d_2 b_2$ of (2).* □

Proof of Lemma 3. Here we deduce relations $\varphi(11)-\varphi(23)$ of the semigroup BT among the words in the alphabet $\mathbb{A}$ from relations (1)-(10) and (25)-(40). We use generalized equivalences $(33')-(40')$ from Claim 5. By the star $\star$ we denote the following images of the map φ for $k=1$:

$$(24')\ \varphi(\xi_1)=d_2c_2,\ \varphi(\eta_1)=a_2b_2,\ \varphi(\sigma_1)=b_1d_2d_1b_2,\ \varphi(\sigma_1^{-1})=d_2b_1b_2d_1,\ \varphi(\tau_1)=d_2x_2b_2.$$

Then by (24) we have $\varphi(u_l)=d_2^{l-1}\star b_2^{l-1}$. Note that the words $d_2^l\star b_2^l\in W$ are 1-balanced and 2-balanced (Fig. 5). Then relations $\varphi(11)-\varphi(14)$ are proved by the same scheme:

$$(11)\ \varphi(\xi_1u_l)\overset{(24)}{=}(d_2c_2)(d_2^{l-1}\star b_2^{l-1})\overset{(4)}{\sim}d_2^2(b_2c_2)(d_2^{l-1}\star b_2^{l-1})\overset{(34')}{\sim}d_2^2(d_2^{l-1}\star b_2^{l-1})(b_2c_2)\overset{(4)}{\sim}\varphi(u_{l+2}\xi_1);$$
$$(12)\ \varphi(\eta_1u_l)\overset{(24)}{=}(a_2b_2)(d_2^{l-1}\star b_2^{l-1})\overset{(4)}{\sim}(a_2d_2)(d_2^{l-3}\star b_2^{l-3})b_2^2\overset{(36')}{\sim}(d_2^{l-3}\star b_2^{l-3})(a_2d_2)b_2^2\overset{(4)}{\sim}\varphi(u_{l-2}\eta_1);$$
$$(13)\ \varphi(\sigma_1u_l)\overset{(24)}{=}(b_1d_2d_1b_2)(d_2^{l-1}\star b_2^{l-1})\overset{(4)}{\sim}(b_1d_2d_1)(d_2^{l-2}\star b_2^{l-1})\overset{(25),(2)}{\sim}d_2^2(b_2d_0d_2b_0)(d_2^{l-3}\star b_2^{l-3})b_2^2\overset{(37')}{\sim}$$
$$\overset{(37')}{\sim}d_2^2(d_2^{l-3}\star b_2^{l-3})(b_2d_0d_2b_0)b_2^2\overset{(4),(26)}{\sim}(d_2^{l-1}\star b_2^{l-2})(b_2b_1)d_2d_1b_2\overset{(24)}{=}\varphi(u_l\sigma_1);$$
$$(14)\ \varphi(\tau_1u_l)\overset{(24)}{=}(d_2x_2b_2)(d_2^{l-1}\star b_2^{l-1})\overset{(4)}{\sim}d_2^2(b_2x_2d_2)(d_2^{l-3}\star b_2^{l-3})b_2^2\overset{(39')}{\sim}d_2^2(d_2^{l-3}\star b_2^{l-3})(b_2x_2d_2)b_2\overset{(4)}{\sim}$$
$$\overset{(4)}{\sim}(d_2^{l-1}\star b_2^{l-1})(d_2x_2b_2)\overset{(24)}{=}\varphi(u_l\tau_1).$$

The remaining calculations are straightforward:

$$(15)\ \varphi(\eta_2\xi_1)\overset{(24)}{=}(d_2a_2b_2^2)(d_2c_2)\overset{(4)}{\sim}d_2(a_2b_2c_2)\overset{(30)}{\sim}d_2b_2\overset{(4)}{\sim}1\overset{(4)}{\sim}d_2b_2\overset{(30)}{\sim}(a_2d_2c_2)b_2\overset{(4)}{\sim}\varphi(\eta_1\xi_2);$$

$$(16)\ \begin{cases}\varphi(\eta_3\sigma_3\xi_1)\overset{(24),(4)}{\sim}d_2^2a_2b_2(b_2b_1)d_2d_1(b_2c_2)\overset{(26)}{\sim}d_2^2(a_2b_2)d_0d_2d_1(b_2c_2)\overset{(35)}{\sim}d_2^2d_0(a_2b_2)d_2d_1(b_2c_2)\overset{(34)}{\sim}\\ \overset{(34)}{\sim}d_2(d_2d_0)(a_2b_2)d_2(b_2c_2)d_1\overset{(4)}{\sim}d_2(d_2d_0)(a_2b_2c_2)d_1\overset{(25)}{\sim}d_2b_1(a_2b_2c_2)d_1\overset{(30)}{\sim}d_2b_1b_2d_1b_2\overset{(24)}{=}\varphi(\sigma_1^{-1}),\\ \\ \varphi(\eta_1\sigma_2\xi_3)\overset{(4)}{\sim}a_2b_1d_2(d_1d_2)c_2b_2^2\overset{(25)}{\sim}a_2b_1d_2(b_0c_2)b_2^2\overset{(1)}{\sim}(a_0d_1)b_1d_2c_1b_2^2\overset{(4)}{\sim}a_0d_2(b_1d_1)c_1b_2^2\overset{(33)}{\sim}\\ \overset{(33)}{\sim}a_0d_2b_1b_2(d_1c_1)b_2\overset{(37)}{\sim}(d_2b_1b_2d_1)(a_0c_1)b_2\overset{(1)}{\sim}d_2b_1b_2d_1d_2b_2\overset{(4)}{\sim}d_2b_1b_2d_1\overset{(24)}{=}\varphi(\sigma_1^{-1});\end{cases}$$

$$(17)\ \varphi(\eta_3\tau_3\xi_1)\overset{(24),(4)}{\sim}d_2^2a_2b_2x_2b_2c_2\overset{(5)}{\sim}d_2^2(b_2x_2b_2)\overset{(4)}{\sim}\varphi(\tau_1)\overset{(4)}{\sim}(d_2x_2d_2)b_2^2\overset{(5)}{\sim}a_2d_2x_2d_2c_2b_2^2\overset{(4)}{\sim}\varphi(\eta_1\tau_2\xi_3);$$

$$(18)\ \begin{cases}\varphi(\sigma_1\xi_1)\overset{(24)}{=}(b_1d_2d_1b_2)(a_2b_2)\overset{(4)}{\sim}b_1d_2(d_1c_2)\overset{(28)}{\sim}b_1(d_2c_0)\overset{(28)}{\sim}b_1c_1\overset{(25)}{\sim}(d_2d_0)c_1\overset{(28)}{\sim}a_2b_2\overset{(24)}{=}\varphi(\xi_1);\\ \varphi(\eta_1\sigma_1)\overset{(4)}{\sim}a_2(b_2b_1)d_2d_1b_2\overset{(26)}{\sim}(a_2d_0)d_2d_1b_2\overset{(1)}{\sim}(a_1d_2)d_1b_2\overset{(1)}{\sim}(a_0d_1)b_2\overset{(1)}{\sim}\varphi(\eta_k);\end{cases}$$

$$(19)\ \varphi(\sigma_1\sigma_1^{-1})\overset{(24)}{=}(b_1d_2d_1b_2)(d_2b_1b_2d_1)\overset{(4)}{\sim}(b_1d_2d_1)(b_1b_2d_1)\overset{(4)}{\sim}1\overset{(4)}{\sim}(d_2b_1b_2d_1)(b_1d_2d_1b_2)\overset{(24)}{=}\varphi(\sigma_1^{-1}\sigma_1);$$

$$(20)\ \begin{cases}\varphi(\sigma_2\sigma_1\sigma_2)\overset{(4)}{\sim}d_2b_1d_2d_1b_2^2b_1d_2^2d_1b_2^2\overset{(26)}{\sim}d_2(b_1d_2d_1b_2)d_0d_2^2d_1b_2^2\overset{(37)}{\sim}d_2d_0(b_1d_2d_1b_2)d_2^2d_1b_2^2\overset{(25),(4)}{\sim}\\ \overset{(25),(4)}{\sim}b_1^2d_2(d_1d_2)d_1b_2^2\overset{(26)}{\sim}b_1^2d_2(d_1d_2)d_1(d_0d_1)b_2\overset{(25)}{\sim}b_1^2d_2b_0d_1(d_0d_1)b_2\overset{(4)}{\sim}b_1^2(d_2b_0d_1d_0b_2)d_2d_1b_2\overset{(40)}{\sim}\\ \overset{(40)}{\sim}b_1^2(b_0d_2d_1b_2d_0)d_2d_1b_2\overset{(25)}{\sim}b_1^2(d_1d_2)d_2d_1b_2(b_2b_1)d_2d_1b_2\overset{(26)}{\sim}b_1d_2^2d_1b_2(b_2b_1)d_2d_1b_2\overset{(4)}{\sim}\varphi(\sigma_1\sigma_2\sigma_1);\end{cases}$$

$$(21)\ \begin{cases}\varphi(\tau_2\sigma_1\sigma_2)\overset{(4)}{\sim}d_2^2x_2b_2^2b_1d_2^2d_1b_2^2\overset{(26)}{\sim}d_2(d_2x_2b_2)d_0d_2^2d_1b_2^2\overset{(38)}{\sim}d_2d_0(d_2x_2b_2)d_2^2d_1b_2^2\overset{(4)}{\sim}\\ \overset{(4)}{\sim}(d_2d_0)(d_2x_2d_2)d_1b_2^2\overset{(25)}{\sim}b_1(d_2x_2d_2)d_1b_2^2\overset{(4)}{\sim}b_1d_2^2(b_2x_2d_2)d_1b_2^2\overset{(39)}{\sim}b_1^2d_2^2d_1(b_2x_2d_2)b_2^2\overset{(4)}{\sim}\varphi(\sigma_1\sigma_2\tau_1).\end{cases}$$

$$(22)\ \begin{cases}\varphi(\tau_1\sigma_2\sigma_1)\overset{(4)}{\sim}d_2x_2(b_1d_2d_1b_2)(b_2b_1)d_2d_1b_2\overset{(26)}{\sim}d_2x_2(b_1d_2d_1b_2)d_0d_2d_1b_2\overset{(37),(4)}{\sim}\\ \overset{(37),(4)}{\sim}d_2(d_0b_0)x_2d_0(b_1d_2d_1^2)b_2\overset{(31)}{\sim}d_2d_0x_1(b_1d_2d_1^2)b_2\overset{(4)}{\sim}d_2d_0b_1(d_1x_1b_1)d_2d_1^2b_2\overset{(38)}{\sim}\\ \overset{(38)}{\sim}d_2d_0b_1d_2(d_1x_1b_1)d_1^2b_2\overset{(2),(4)}{\sim}d_2d_0b_1d_2d_1(b_0x_2d_0)d_1b_2\overset{(25),(4)}{\sim}d_2d_0(b_1d_2d_1b_2)d_2b_0x_2b_2^2\overset{(37)}{\sim}\\ \overset{(37)}{\sim}d_2(b_1d_2d_1b_2)d_0d_2b_0x_2b_2^2\overset{(25),(26)}{\sim}d_2b_1d_2d_1b_2(b_2b_1)d_2(d_1d_2)x_2b_2^2\overset{(4)}{\sim}\varphi(\sigma_2\sigma_1\tau_2);\end{cases}$$

$$(23)\ \varphi(\sigma_1\tau_1) \overset{(4)}{\sim} b_1d_2d_1x_2b_2 \overset{(25)}{\sim} (d_2d_0)d_2d_1x_2b_2 \overset{(6)}{\sim} d_2x_2(d_0d_2d_1)b_2 \overset{(26)}{\sim} d_2x_2(b_2b_1)d_2d_1b_2 \overset{(4)}{\sim} \varphi(\tau_1\sigma_1).$$

$\square$

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